A circular distribution family and tests for independence

Marcus Vollmer

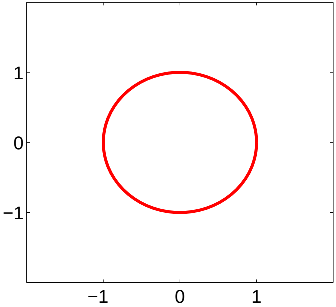
Department of Mathematics and Computer Science
University of Greifswald

COPULÆ IN MATHEMATICAL and QUANTITATIVE FINANCE

Kraków
07-11-2012

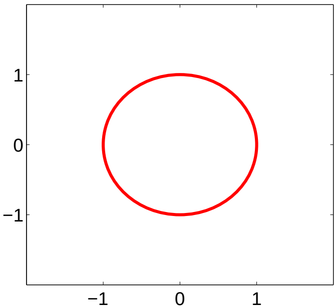
The Circular Uniform Distribution

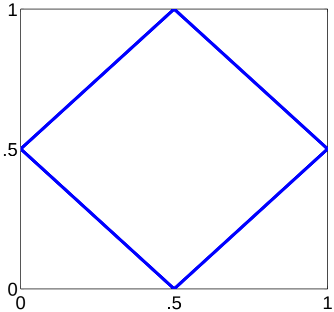
Nelsen: $\Theta \sim U(0, 2\pi)$, $\rho = 1$



The Circular Uniform Distribution

Nelsen: $\Theta \sim U(0, 2\pi)$, $\rho = 1$

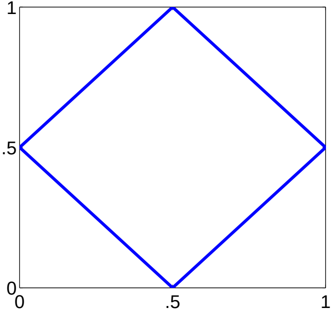




The Circular Uniform Distribution

Nelsen: $\Theta \sim U(0, 2\pi)$, $\rho = 1$

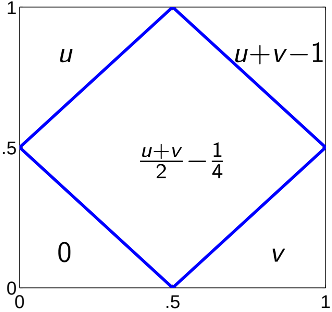
$$C(u,v) = \begin{cases} M(u,v), & |u-v| > \frac{1}{2}, \\ W(u,v), & |u+v-1| > \frac{1}{2}, \\ \frac{u+v}{2} - \frac{1}{4}, & \text{otherwise.} \end{cases}$$



The Circular Uniform Distribution

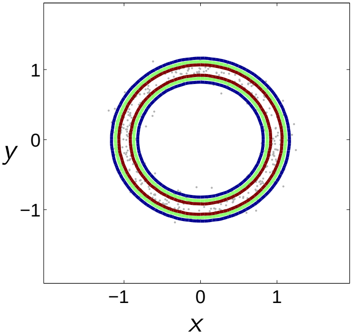
Nelsen: $\Theta \sim U(0, 2\pi)$, $\rho = 1$

$$C(u,v) = \begin{cases} M(u,v), & |u-v| > \frac{1}{2}, \\ W(u,v), & |u+v-1| > \frac{1}{2}, \\ \frac{u+v}{2} - \frac{1}{4}, & \text{otherwise.} \end{cases}$$



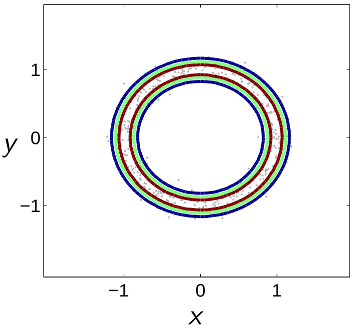
The Circular Distribution Family

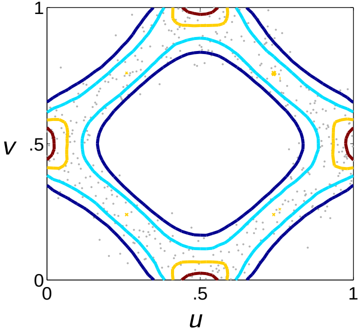
Vollmer: $\Theta \sim U(0, 2\pi)$, $\rho \sim N(1, \sigma^2)$,



The Circular Distribution Family

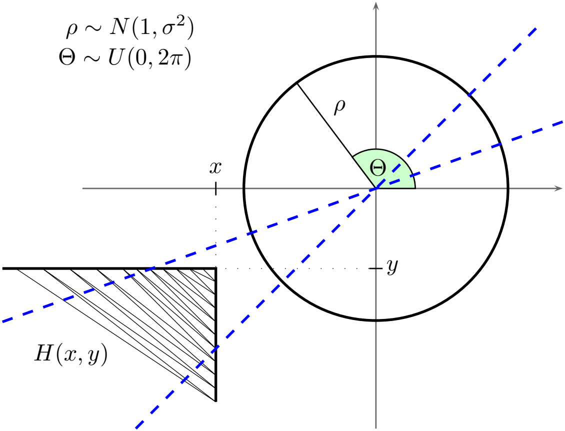
Vollmer: $\Theta \sim U(0, 2\pi)$, $\rho \sim N(1, \sigma^2)$,



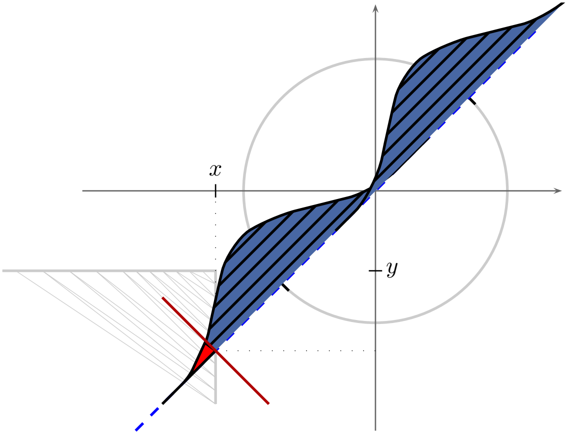


$\sigma = 0.00..0.37$

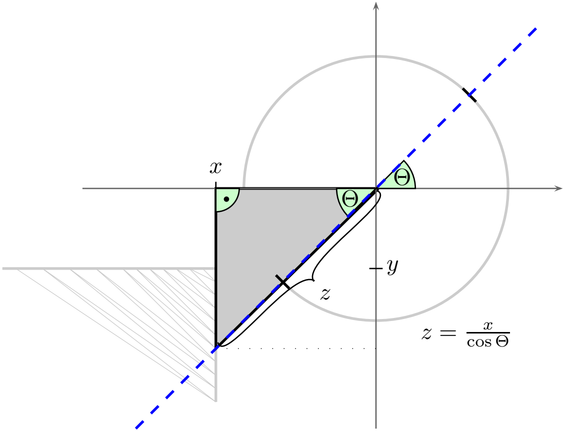
Geometry on the Circle Distribution



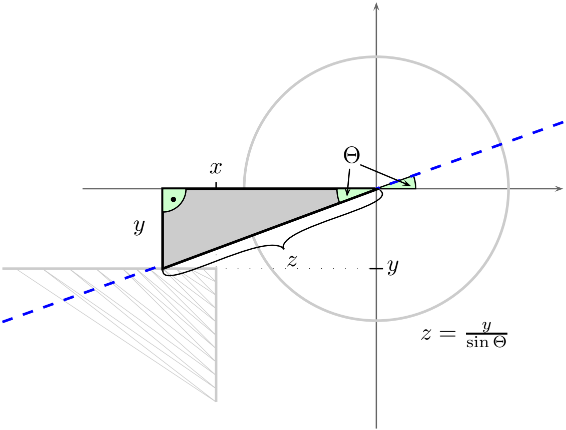
Geometry on the Circle Distribution



Geometry on the Circle Distribution



Geometry on the Circle Distribution



CDF case: $x, y < 0$

$$H_\sigma(x,y) = \frac{1}{2\pi}\int_0^{\cot\left(\frac{y}{x}\right)} \Phi\left(\frac{1}{\sigma}\left(\frac{y}{\sin\Theta} - 1\right)\right) + \Phi\left(\frac{1}{\sigma}\left(\frac{y}{\sin\Theta} + 1\right)\right) \mathrm{d}\Theta + \frac{1}{2\pi}\int_{\cot\left(\frac{y}{x}\right)}^{\frac{\pi}{2}} \Phi\left(\frac{1}{\sigma}\left(\frac{x}{\cos\Theta} - 1\right)\right) + \Phi\left(\frac{1}{\sigma}\left(\frac{x}{\cos\Theta} + 1\right)\right) \mathrm{d}\Theta$$

CDF case: $x, y < 0$

Numerical calculation:

choose equidistant segmentation of $\Theta \in [0, 2\pi]$ with step size d

$$H_\sigma(x,y) = \frac{1}{2\pi}\int_0^{\cot\left(\frac{y}{x}\right)} \Phi\left(\frac{1}{\sigma}\left(\frac{y}{\sin\Theta} - 1\right)\right) + \Phi\left(\frac{1}{\sigma}\left(\frac{y}{\sin\Theta} + 1\right)\right) \mathrm{d}\Theta + \frac{1}{2\pi}\int_{\cot\left(\frac{y}{x}\right)}^{\frac{\pi}{2}} \Phi\left(\frac{1}{\sigma}\left(\frac{x}{\cos\Theta} - 1\right)\right) + \Phi\left(\frac{1}{\sigma}\left(\frac{x}{\cos\Theta} + 1\right)\right) \mathrm{d}\Theta$$

CDF case: $x, y < 0$

Numerical calculation:

choose equidistant segmentation of $\Theta \in [0, 2\pi]$ with step size d

$$H_\sigma(x,y) \approx \frac{2\pi}{d} \sum_{\Theta=0,d,\ldots,\cot\left(\frac{y}{x}\right)} \left[\Phi\left(\frac{1}{\sigma}\left(\frac{y}{\sin\Theta} - 1\right)\right) + \Phi\left(\frac{1}{\sigma}\left(\frac{y}{\sin\Theta} + 1\right)\right) \right] +$$

$$\frac{2\pi}{d} \sum_{\Theta=\cot\left(\frac{y}{x}\right),\ldots,\frac{\pi}{2}} \left[\Phi\left(\frac{1}{\sigma}\left(\frac{x}{\cos\Theta} - 1\right)\right) + \Phi\left(\frac{1}{\sigma}\left(\frac{x}{\cos\Theta} + 1\right)\right) \right]$$

CDF: all cases

Other cases are as easy as for $x, y \leq 0$.

In general I have a MATLAB function `circfamcdf(x,y,sigma)` based on segmentation and using the trigonometric functions.

The marginal cdfs $F_\sigma(y)$ and $G_\sigma(x)$ are due to the limits $y \to \infty$ or $x \to \infty$ of $H_\sigma(x, y)$ and

$$F_\sigma(y) = G_\sigma(x) = \lim_{y \to \infty} H_\sigma(x, y)$$

Calculation of $C(u, v)$

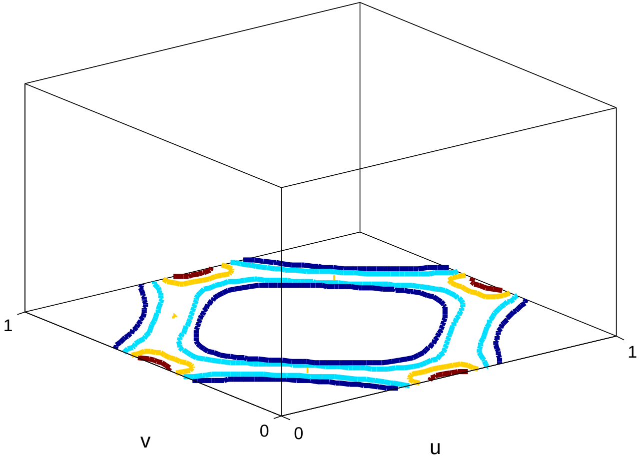
$$C_\sigma(u, v) = H_\sigma(F_\sigma^{-1}(u), F_\sigma^{-1}(v))$$

Instruction

1. Estimate $H_\sigma(x, y)$ (Segmentation / Trigonometric funct.)
2. Compute $F_\sigma = \lim_{y\to\infty} H(x, y)$
3. Estimate the inverse function of F (`fzero`-Method)
4. Compute $C_\sigma(u, v)$ on a grid $\in [0, 1] \times [0, 1]$

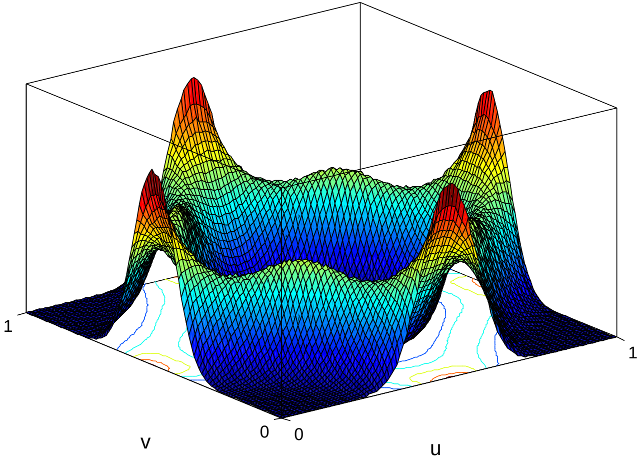
The Copula of the Circular Distribution Family

$\sigma = 0.1$



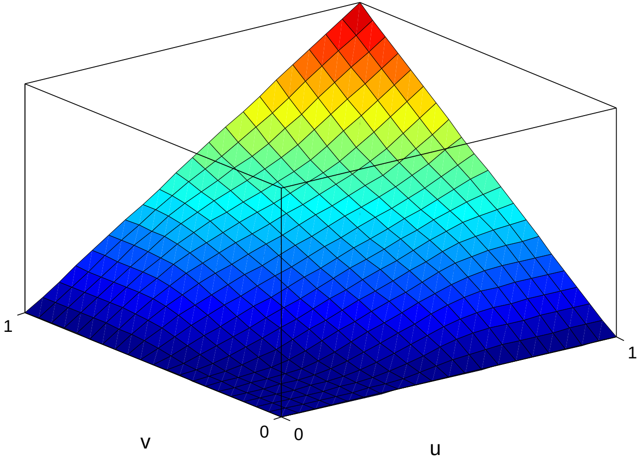
The Copula of the Circular Distribution Family

$\sigma = 0.1$



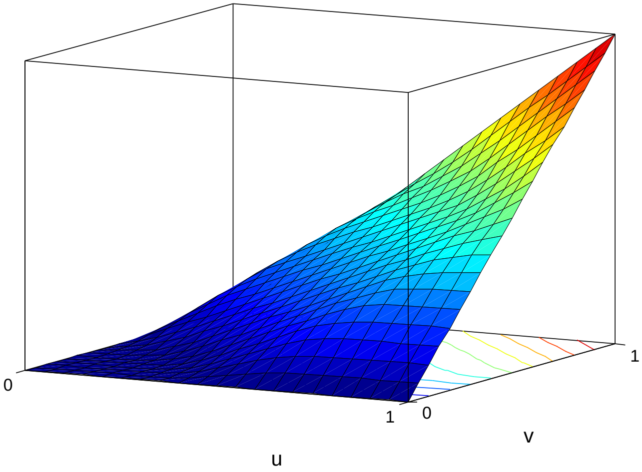
The Copula of the Circular Distribution Family

$\sigma = 0.1$

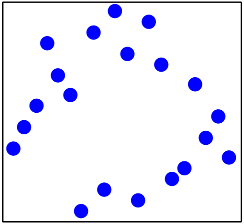


The Copula of the Circular Distribution Family

$\sigma = 0.1$

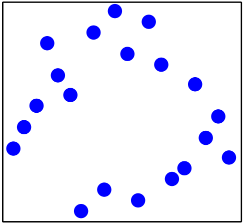


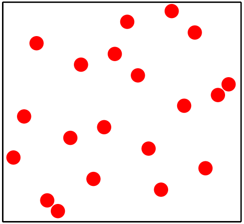
How to test for independence?



Sample of the
Circular distribution family

How to test for independence?



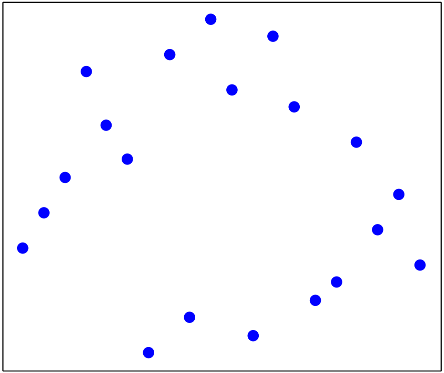


Sample of the
Circular distribution family

Independent sample
Uniform distribution

How to test for independence?

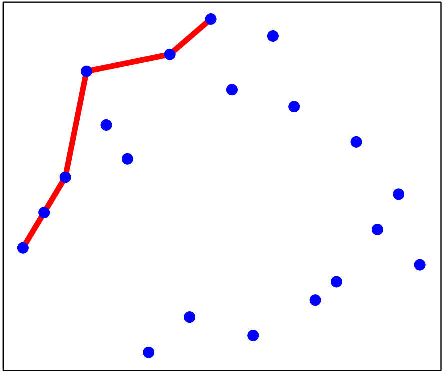
Recent ideas: LIS



How to test for independence?

Recent ideas: LIS

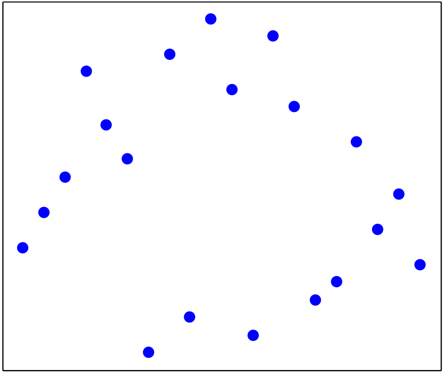
Test statistic based on
Longest **I**ncreasing **S**ubsequence



J. E. García, V. A. González-López, *A Nonparametric Independence Test using Random Permutations*, Preprint, arXiv:0908.2794v2, 2009.

How to test for independence?

Recent ideas: PompeLE

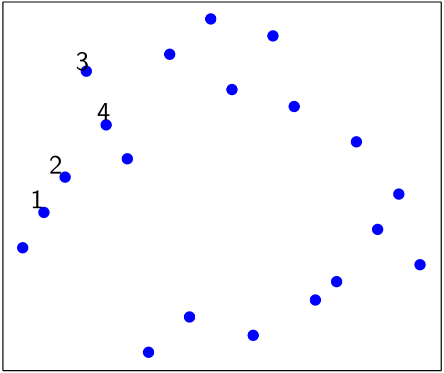


How to test for independence?

Recent ideas: PompeLE

Test statistic based on distance to k^{th} successor

$k = \lfloor \sqrt{n} \rfloor = 4$



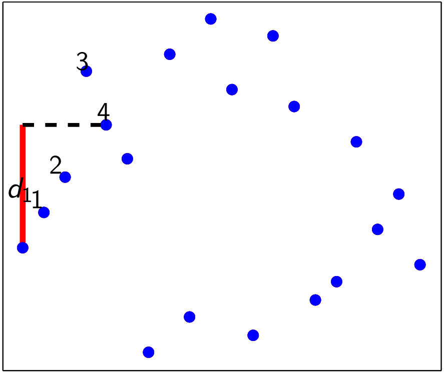
B. Pompe, *The LE-Statistic: A Versatile Tool in Ordinal Time Series Analysis*,
Lecture at 9th AIMS, July 1-5, 2012, Orlando, Florida.

How to test for independence?

Recent ideas: PompeLE

Test statistic based on distance to k^{th} successor

$k = \lfloor \sqrt{n} \rfloor = 4$



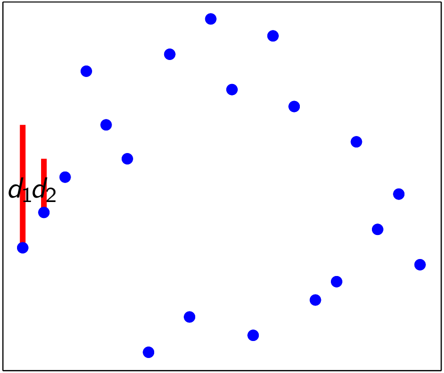
B. Pompe, *The LE-Statistic: A Versatile Tool in Ordinal Time Series Analysis*,
Lecture at 9th AIMS, July 1-5, 2012, Orlando, Florida.

How to test for independence?

Recent ideas: PompeLE

Test statistic based on distance to k^{th} successor

$k = \lfloor \sqrt{n} \rfloor = 4$



B. Pompe, *The LE-Statistic: A Versatile Tool in Ordinal Time Series Analysis*,
Lecture at 9th AIMS, July 1-5, 2012, Orlando, Florida.

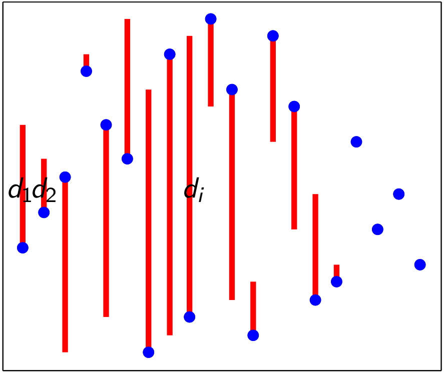
How to test for independence?

Recent ideas: PompeLE

Test statistic based on distance to k^{th} successor

$k = \lfloor \sqrt{n} \rfloor = 4$

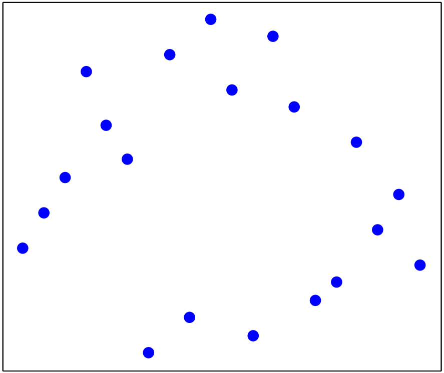
$\mathrm{L}_{\mathrm{yx}} = \sum_{i=1}^{n-k} \log d_i$



B. Pompe, *The LE-Statistic: A Versatile Tool in Ordinal Time Series Analysis*,
Lecture at 9th AIMS, July 1-5, 2012, Orlando, Florida.

How to test for independence?

Recent ideas: kNN

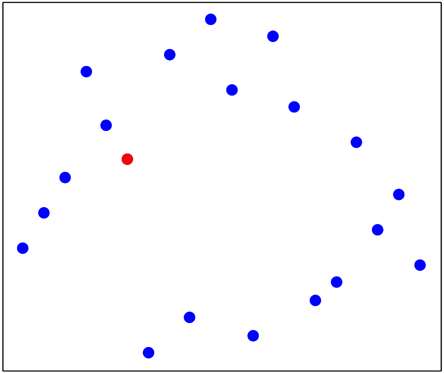


How to test for independence?

Recent ideas: kNN

Test statistic based on the distance to the **k**$^{\text{th}}$ **n**earest **n**eighbour

$k = \lfloor \sqrt{n} - \frac{3}{2} \rfloor = 2$

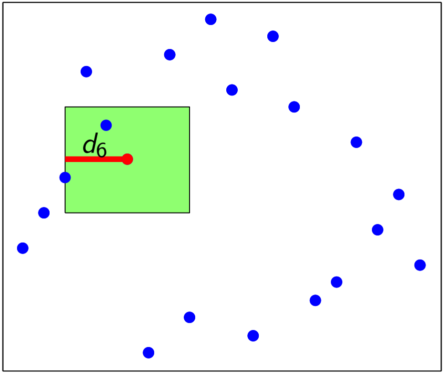


How to test for independence?

Recent ideas: kNN

Test statistic based on the distance to the **k**th **n**earest **n**eighbour

$k = \lfloor \sqrt{n} - \frac{3}{2} \rfloor = 2$

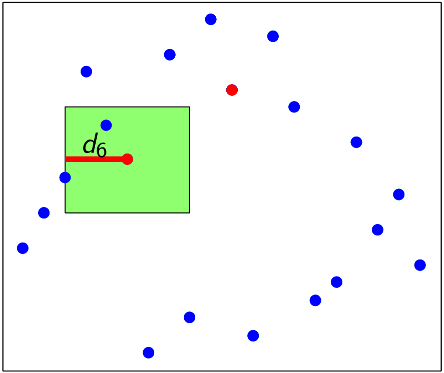


How to test for independence?

Recent ideas: kNN

Test statistic based on the distance to the **k**th **n**earest **n**eighbour

$k = \lfloor \sqrt{n} - \frac{3}{2} \rfloor = 2$

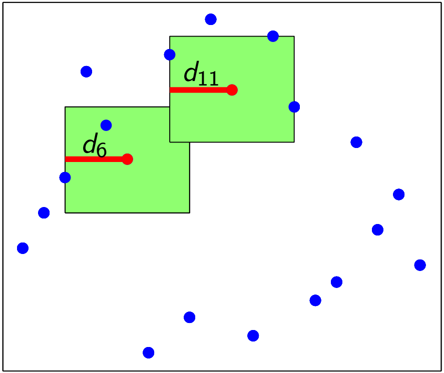


How to test for independence?

Recent ideas: kNN

Test statistic based on the distance to the **k**th **n**earest **n**eighbour

$k = \lfloor \sqrt{n} - \frac{3}{2} \rfloor = 2$



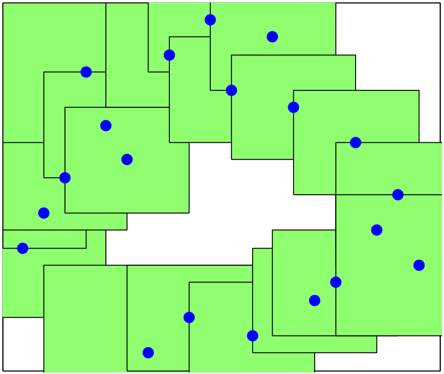
How to test for independence?

Recent ideas: kNN

Test statistic based on the distance to the **k**th **n**earest **n**eighbour

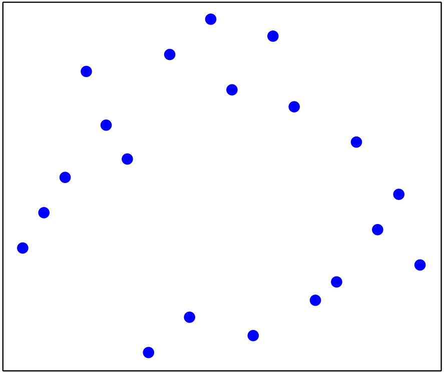
$k = \lfloor \sqrt{n} - \frac{3}{2} \rfloor = 2$

$S = \sum_{i=1}^{n} d_i^2$



How to test for independence?

Recent ideas: GRaP

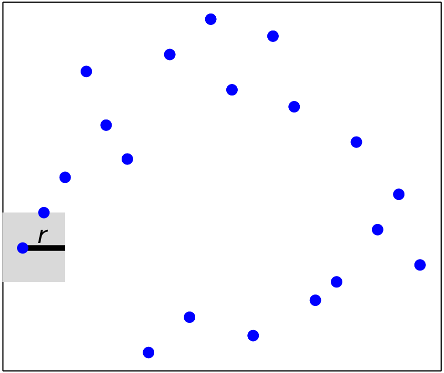


How to test for independence?

Recent ideas: GRaP

Test statistic based on counting coordinates overlapped by squares

$r = \lfloor \sqrt{n} - \frac{3}{2} \rfloor = 2$



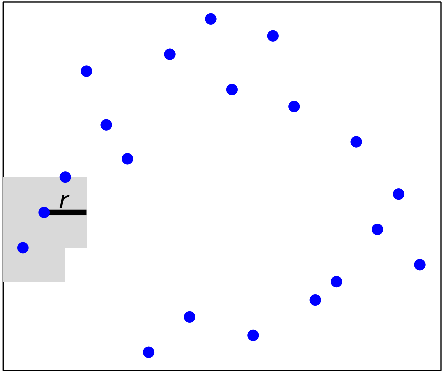
M. Vollmer, *A new Independence Test for continuous variables*, Talk at ERCIM'11, December 19, 2011, London, UK.

How to test for independence?

Recent ideas: GRaP

Test statistic based on counting coordinates overlapped by squares

$r = \lfloor \sqrt{n} - \frac{3}{2} \rfloor = 2$



M. Vollmer, *A new Independence Test for continuous variables*, Talk at ERCIM'11, December 19, 2011, London, UK.

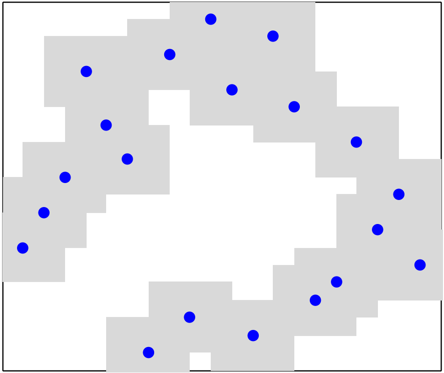
How to test for independence?

Recent ideas: GRaP

Test statistic based on counting coordinates overlapped by squares

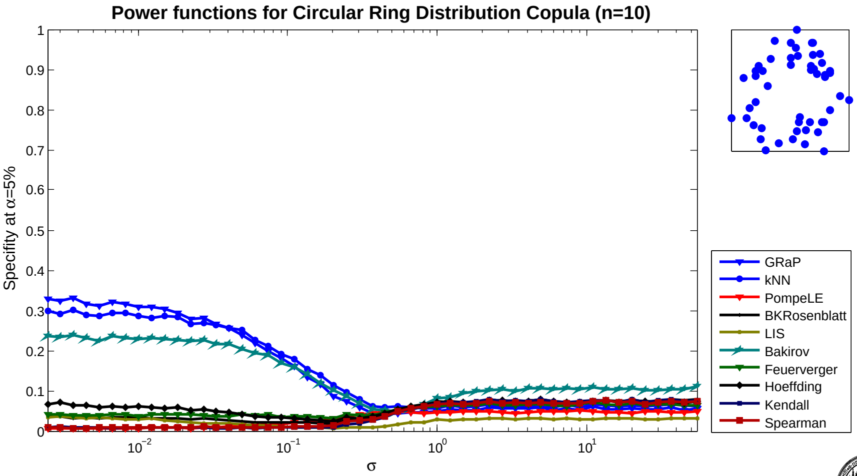
$r = \lfloor \sqrt{n} - \frac{3}{2} \rfloor = 2$

$S =$ Number of (i, j) covered by grey area



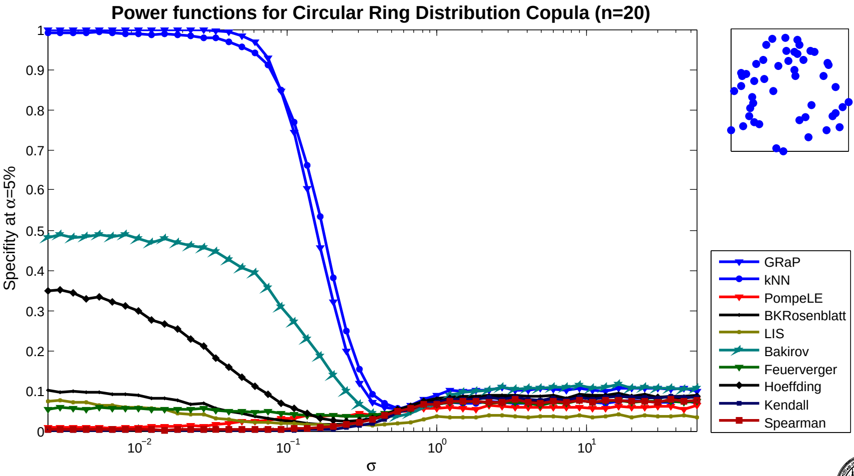
M. Vollmer, *A new Independence Test for continuous variables*, Talk at ERCIM'11, December 19, 2011, London, UK.

Power functions



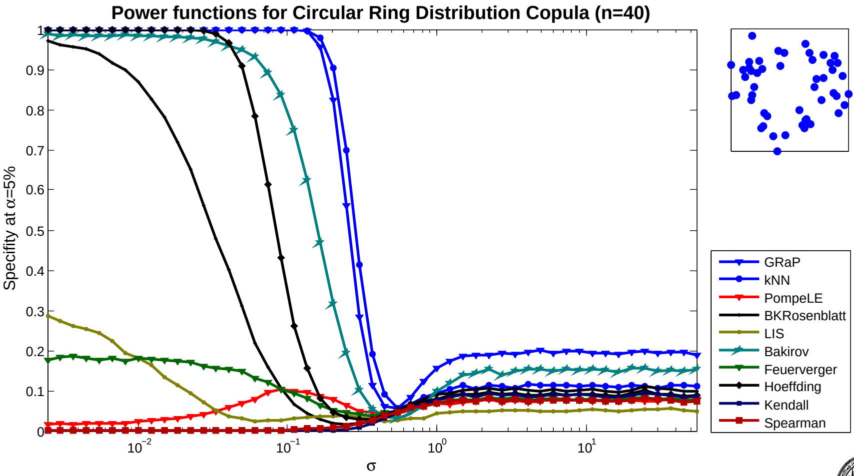
skip

Power functions



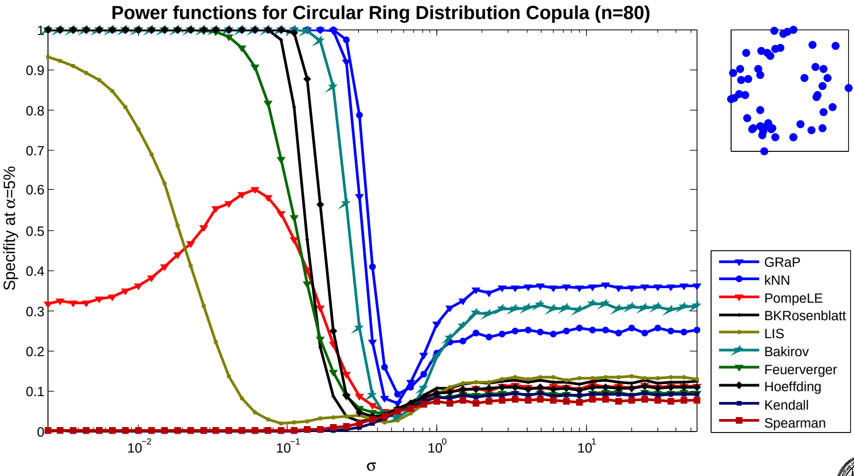
skip

Power functions



skip

Power functions



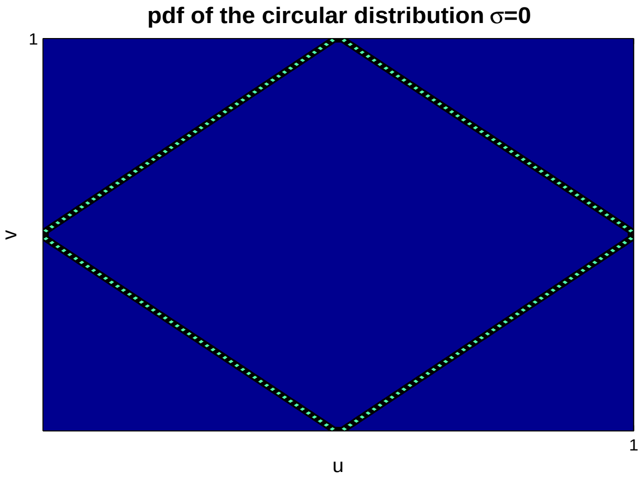
skip

Discussion

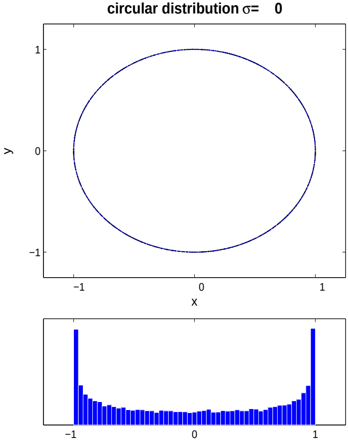
Thank you for your kind attention!

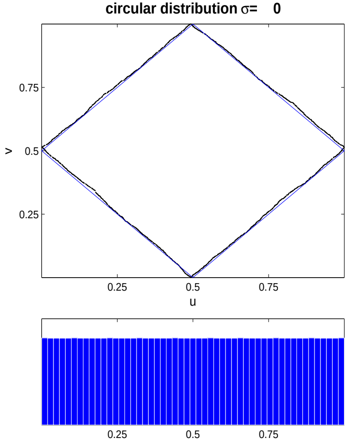
R. B. Nelsen, *An Introduction to Copulas*,
ISBN: 978-0-387-28659-4 ,Springer, 2nd. Edition, 2006.

Contour plot of the Circular Distribution Family



Scatterplot animation of the family





Marginal cdf of the Circular Distribution Family